

# A NEW STOCHASTIC ESTIMATOR FOR TREMOR FREQUENCY TRACKING

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## ABSTRACT

An important parameter in analysis of physiological tremor is the diagnosis and study of neurological disorders. The instantaneous tremor frequency (ITF) is an important parameter in tremor analysis. This paper proposes a novel stochastic filter, the multiple extended Kalman filter (M-EKF), for tracking of ITF from neural microelectrode recordings. The M-EKF mitigates degradations in filter performance resulting from a mismatch between assumed initial conditions and those of a particular realization of a stochastic system. Specifically, the M-EKF is comprised of a bank of extended Kalman filters (EKF), each initialized with different conditions, selected according to the *unscented transform*. The final estimate is a weighted average of the individual estimates provided by each EKF where the weights reflect how closely the assumed EKF initial conditions match those of the true system. The M-EKF is applied to a synthetic tremor model to display its superior performance to that of the EKF and the unscented Kalman filter.

**Index Terms**— Tremor frequency, state-space model, nonlinear estimation, unscented transform, extended Kalman filtering.

## 1. INTRODUCTION

Tremor is an involuntary, quasi-periodic, oscillation of one or more muscles of the body, often expressed as a symptom of various neurological diseases [1]. Tremor activity is therefore a principal aspect of the diagnosis and study of neurological movement disorders. This paper addresses the problem of *tracking* or continuous estimation of the instantaneous tremor frequency (ITF), an important parameter in tremor analysis [2].

The work of [3] proposes the use of the *stochastic filtering* framework for ITF tracking from binary spike trains observed in neural microelectrode recordings (MER). In this approach, domain knowledge, such as dynamics of the system and statistical characteristics of observation noise, are incorporated into the estimation process through a *state-space* description of the system. The following *state-space* model is considered herein

$$\begin{aligned}\mathbf{x}(k) &= g(\mathbf{x}(k-1)) + \mathbf{u}(k), \\ \mathbf{z}(k) &= h(\mathbf{x}(k)) + \mathbf{v}(k),\end{aligned}\quad (1)$$

where  $\mathbf{x}(k)$  is a *state vector* containing all variables required to describe system dynamics,  $\mathbf{z}(k)$  is the observed *measurement vector*,  $\mathbf{u}(k) \sim \mathcal{N}(0, \mathbf{Q}(k))$  and  $\mathbf{v}(k) \sim \mathcal{N}(0, \mathbf{R}(k))$  are white Gaussian process and measurement noise, respectively. The functions  $g(\cdot)$  and  $h(\cdot)$  are nonlinear and real valued, and  $k$  is the discrete time

index. The filter estimate at time step  $k$  given the measurement set  $\{\mathbf{z}(1), \dots, \mathbf{z}(j)\}$  is denoted as  $\hat{\mathbf{x}}(k|j)$ .

Given this *state-space* formulation, well-known stochastic filters can be used to estimate the unobservable state vector from noisy measurements over time [4]. When the system and measurement equations are both linear and noise distributions are Gaussian, the Kalman filter provides the optimal estimate in the minimum mean square error (MMSE) sense. In the case of nonlinear system or measurement equations, the optimal estimate cannot be directly determined and the extended Kalman filter (EKF) [4], unscented Kalman filter (UKF) [5], and the particle filter [6] are commonly used to provide suboptimal solutions. Since ITF is nonlinearly related to the measurements obtained from neural recordings, the work of [3] uses an *extended Kalman smoother* as the tracking solution.

The model of (1) describes the evolution of a stochastic system in a recursive manner and therefore requires specification of initial conditions. Due to the stochastic nature of the system, resulting from the presence of noise and modeling inaccuracies, the initial state of the system is assumed to be random and generally modeled as a Gaussian random variable with distribution  $\mathcal{N}(\hat{\mathbf{x}}(0|0), \mathbf{P}(0|0))$ . The mean value  $\hat{\mathbf{x}}(0|0)$  is used as the initial state in the Kalman filter and its variants. However, discrepancies between the assumed conditions by the filter and the true realization of state lead to a degradation in accuracy of the estimates computed by the filter. To mitigate such adverse effects, this paper proposes a novel stochastic filter, the multiple-EKF (M-EKF) for nonlinear estimation problems such as ITF tracking. As shown in Figure 1, the M-EKF is comprised of a bank of EKFs running in parallel, initialized with a different set of initial conditions. The estimates provided by these filters are then fused using a weighted average where the weights are proportional to how well the respective initial conditions represent the true conditions.

The rest of this paper is organized as follows. Section 2 discusses the effect of initial conditions, Section 3 introduces the details of the M-EKF algorithm, Section 4 applies the M-EKF to the problem of ITF tracking, and Section 5 concludes the paper and provides directions for future work.

## 2. EFFECT OF INITIAL CONDITIONS ON THE EKF

To motivate the development of the M-EKF, this section examines the effect of initial conditions on the EKF. The EKF [4] uses a first order Taylor series expansion of the functions  $g(\cdot)$  and  $h(\cdot)$  in (1) and (2) to obtain a linear approximation to the *state-space* description of the system. Define the Jacobians  $\mathbf{G}(k)$  and  $\mathbf{H}(k)$  as follows

$$\mathbf{G}(k) = \left. \frac{\partial g}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}(k-1|k-1)} \quad \text{and} \quad \mathbf{H}(k) = \left. \frac{\partial h}{\partial \mathbf{x}} \right|_{\hat{\mathbf{x}}(k|k-1)} \quad (3)$$

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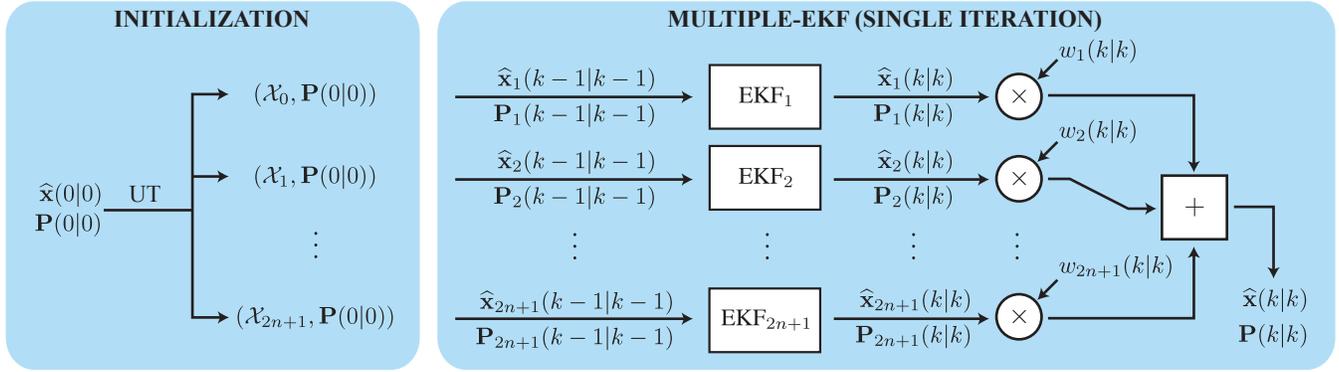


Fig. 1. Multiple-EKF System Level Operation.

Then, the approximated linear version of the *state-space* model (1) and (2) is given as

$$\mathbf{x}(k) = \mathbf{G}(k)\mathbf{x}(k-1) + \mathbf{u}(k), \quad (4)$$

$$\mathbf{z}(k) = \mathbf{H}(k)\mathbf{x}(k) + \mathbf{v}(k). \quad (5)$$

The extended Kalman filter [4], shown in Figure 2, can now be used with the linearized system in (4) and (5). Note that while the Kalman filter provides the optimal MMSE estimate in the case of linear and Gaussian systems, the EKF estimates are suboptimal due to the un-modeled dynamics (UMD) arising from linearization errors. It can be seen that, in the case of the measurement equation (2) the following difference

$$\text{UMD} = h(\hat{\mathbf{x}}(k|k-1)) - \mathbf{H}(k)\hat{\mathbf{x}}(k|k-1) \quad (6)$$

is not represented within the EKF framework.

#### Inputs:

Initial state distribution:  $\mathcal{N}(\hat{\mathbf{x}}(0|0), \mathbf{P}(0|0))$

Measurement record:  $\{\mathbf{z}(1), \dots, \mathbf{z}(K)\}$

#### Outputs:

State estimate at time  $k$ :  $\hat{\mathbf{x}}(k|k)$

Estimation covariance at time  $k$ :  $\mathbf{P}(k|k)$

#### EKF:

##### Prediction

$$\hat{\mathbf{x}}(k|k-1) = g(\hat{\mathbf{x}}(k-1|k-1))$$

$$\mathbf{P}(k|k-1) = \mathbf{G}(k)\mathbf{P}(k-1|k-1)\mathbf{G}(k)^T + \mathbf{Q}(k)$$

##### Update

$$\mathbf{y}(k|k-1) = \mathbf{z}(k) - h(\hat{\mathbf{x}}(k|k-1))$$

$$\mathbf{S}(k|k-1) = \mathbf{H}(k)\mathbf{P}(k|k-1)\mathbf{H}(k)^T + \mathbf{R}(k)$$

$$\mathbf{K}(k) = \mathbf{P}(k|k-1)\mathbf{H}(k)^T\mathbf{S}(k|k-1)^{-1}$$

##### Estimation

$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\mathbf{y}(k|k-1)$$

$$\mathbf{P}(k|k) = (\mathbf{I} - \mathbf{K}(k)\mathbf{H}(k))\mathbf{P}(k|k-1)$$

Fig. 2. Extended Kalman Filter Algorithm.

Now consider the scenario where the initial conditions assumed by the EKF, namely  $\hat{\mathbf{x}}(0|0)$ , is significantly different than the initial conditions of a particular realization of the dynamic stochastic system. We define the criterion for mismatched initial conditions as

$$\|\mathbf{x}(0) - \hat{\mathbf{x}}(0|0)\|^2 \gg \text{trace}\{\mathbf{P}(0|0)\}. \quad (7)$$

The concept of such mismatch is illustrated in Figure 3 for the two dimensional tremor model described in Section 4.1. This figure depicts 1000 points drawn as  $\mathbf{x}(0) \sim \mathcal{N}(\hat{\mathbf{x}}(0|0), \mathbf{P}(0|0))$  indicating the cases where a mismatch in initial conditions occurs.

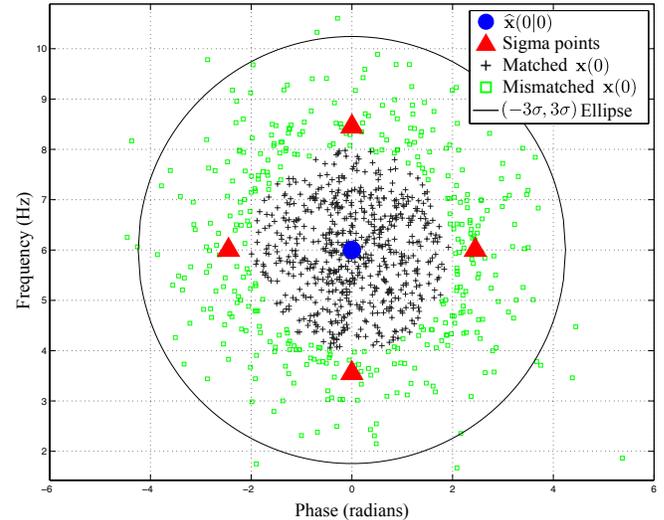


Fig. 3. Distinction between matched and mismatched realizations.

Since the extended Kalman filter performs its initial linearization around the mean  $\hat{\mathbf{x}}(0|0)$ , a mismatch in initial conditions increases the linearization error and the un-modeled system dynamics. Due to the recursive nature of the filter, any errors in the initial state propagate to subsequent estimates and negatively impact the performance of the filter. To mitigate the effect of such errors, the M-EKF uses a bank of EKFs, each performing linearization around a different set of initial conditions. The estimates from these EKFs are then fused together in a weighted average.

### 3. THE M-EKF

#### 3.1. Initial Conditions

The M-EKF is composed of a bank of  $(2n + 1)$  extended Kalman filters running in parallel, where  $n$  is the dimension of the state vec-

tor  $\mathbf{x}(k)$ . Each EKF is initialized with a different set of initial conditions, chosen according to the *unscented transform* (UT) [5]. The UT represents a Gaussian distribution by a set of finite sigma points and associated weights. These sigma points become ideal initial conditions for the bank of EKFs since they cover the Gaussian distribution effectively (see Figure 3).

To generate our set of initial conditions we apply the UT to  $\mathcal{N}(\widehat{\mathbf{x}}(0|0), \mathbf{P}(0|0))$  as follows:

$$\mathcal{X}_0 = \widehat{\mathbf{x}}(0|0) \quad (8)$$

$$\mathcal{X}_i = \widehat{\mathbf{x}}(0|0) + \left( \sqrt{(n + \kappa)\mathbf{P}(0|0)} \right)_i, \quad i = 1, \dots, n \quad (9)$$

$$\mathcal{X}_{i+n} = \widehat{\mathbf{x}}(0|0) - \left( \sqrt{(n + \kappa)\mathbf{P}(0|0)} \right)_i, \quad i = 1, \dots, n \quad (10)$$

where  $\kappa$  is the UT secondary scaling parameter, and  $(\cdot)_i$  denotes the  $i^{\text{th}}$  row of a matrix. It is shown in [5] that  $(n + \kappa) = 3$  performs well for Gaussian distributions.

To initiate the M-EKF, each sigma point  $\mathcal{X}_i$  is paired with the common covariance  $\mathbf{P}(0|0)$ . This couple is used as the initial conditions for one of the  $(2n + 1)$  extended Kalman filters. For example, the third EKF is initialized with  $\mathcal{N}(\mathcal{X}_3, \mathbf{P}(0|0))$ . Consequently, the first EKF is identical to a standard extended Kalman filter, as the first sigma vector defined in (8) is the mean  $\widehat{\mathbf{x}}(0|0)$ .

### 3.2. The M-EKF Algorithm

The bank of extended Kalman filters run in parallel, each generating their own estimates  $\widehat{\mathbf{x}}_i(k|k)$  and  $\mathbf{P}_i(k|k)$ . The estimates are fused through a weighted average at each time step  $k$  to generate the final estimates  $\widehat{\mathbf{x}}(k|k)$  and  $\mathbf{P}(k|k)$ . The algorithm is outlined in Figure 4.

The weights  $w_i(k|k)$  are updated in a linear, autoregressive manner, using the parameter  $\lambda_i(k|k - 1)$ . This value represents the amount of confidence associated with the  $i^{\text{th}}$  EKF instance at time  $k$ . The *innovations* sequence  $\mathbf{y}_i(k|k - 1)$  and covariance  $\mathbf{S}_i(k|k - 1)$  represent the prediction error of each filter, and are utilized to calculate  $\lambda_i(k|k - 1)$  (see Figure 4).

The M-EKF and UKF differ in two respects. First, the UKF utilizes the *unscented transform* weights which are designed such that the first sigma point weighs more than the peripheral sigma points [5]. Since the vicinity of a random realization  $\mathbf{x}(0)$  to any of the sigma points is unknown (see Figure 3), the M-EKF begins by assigning equal weights to  $w_i(0|0)$ .

Second, although the M-EKF employs the *unscented transform* to generate its initial starting vectors  $\mathcal{X}_i$ , it is fundamentally different than the unscented Kalman filter. The *unscented transform* is used only once to initialize the bank of extended Kalman filters within the M-EKF. Each EKF continues by linearizing the *state-space* model according to its own prior estimates and predictions. Contrarily, the UKF attempts to avoid linearization by propagating a deterministically chosen set of sigma points through the nonlinearities of the system [5]. Thus, the *unscented transform* is applied at every time step  $k$ . This approach aims at escaping the Jacobian calculations associated with the EKF, yet the computational complexity of the UKF proceeds to be the same order as that of the EKF [5]. The complexity of the M-EKF is thus  $(2n + 1)$  times that of the EKF.

## 4. RESULTS AND DISCUSSION

### 4.1. Tremor Model

The following tremor model is adopted from the tremor models described in [3], and [7]. It is very similar to the model described in [7],

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#### Inputs:

Initial state distribution:  $\mathcal{N}(\widehat{\mathbf{x}}(0|0), \mathbf{P}(0|0))$

Measurement record:  $\{\mathbf{z}(1), \dots, \mathbf{z}(K)\}$

#### Outputs:

State estimate at time  $k$ :  $\widehat{\mathbf{x}}(k|k)$

Estimation covariance at time  $k$ :  $\mathbf{P}(k|k)$

#### Initial Conditions:

Define initial weights:  $w_i(0|0) = (2n + 1)^{-1}$

#### M-EKF:

Step 1: Choose  $2n + 1$  starting vectors  $\mathcal{X}_i$  (8-10)

Step 2: Initiate  $2n + 1$  extended Kalman filters with starting vectors  $(\mathcal{X}_i, \mathbf{P}(0|0))$

At each time sample  $k$ :

For each EKF:

Calculate  $\widehat{\mathbf{x}}_i(k|k)$  and  $\mathbf{P}_i(k|k)$

Calculate  $\lambda_i(k|k - 1) = \det(\mathbf{S}_i(k|k - 1))^{-\frac{1}{2}} \times \exp\{-\frac{1}{2}\mathbf{y}_i^T(k|k - 1)\mathbf{S}_i^{-1}(k|k - 1)\mathbf{y}_i(k|k - 1)\}$

Calculate  $\Omega_i = \lambda_i(k|k - 1)w_i(k - 1|k - 1)$

Normalize weights as  $w_i(k|k) = \frac{\Omega_i}{\sum_{i=1}^{2n+1}\{\Omega_i\}}$

Merge all  $2n + 1$  estimates:

Calculate  $\widehat{\mathbf{x}}(k|k) = \sum_{i=1}^{2n+1} w_i(k|k)\widehat{\mathbf{x}}_i(k|k)$

Calculate  $\mathbf{P}(k|k) = \sum_{i=1}^{2n+1} w_i(k|k)[\mathbf{P}_i(k|k) + \{\widehat{\mathbf{x}}(k|k) - \widehat{\mathbf{x}}_i(k|k)\}\{\widehat{\mathbf{x}}(k|k) - \widehat{\mathbf{x}}_i(k|k)\}^T]$

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Fig. 4. Multiple-EKF Algorithm.

yet with some elements based on the authors' antecedent paper [3]. The hidden state vector  $\mathbf{x}(k) = [\theta(k) \ f(k)]^T$  contains the instantaneous phase  $\theta(k)$  and frequency  $f(k)$  of the tremor signal. The *state-space* model is described as,

$$\mathbf{x}(k) = \begin{bmatrix} \theta(k - 1) + 2\pi T_s f(k - 1) \bmod 2\pi \\ \gamma(f(k - 1) - \bar{f}) + \bar{f} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad (11)$$

$$\mathbf{z}(k) = a \sin(2\pi T_s \bar{f} k + \theta(k)) + v(k) \quad (12)$$

where both  $u(k) \sim \mathcal{N}(0, q)$  and  $v(k) \sim \mathcal{N}(0, r)$  are zero-mean white Gaussian processes, and the  $\bmod 2\pi$  operation is applied to all of the first row.

Parameter values required for signal generation and M-EKF operation are summarized in Table 1. It is assumed that all parameters are known to the designer of the filter, although the amplitude of the measurements can also be estimated [3].

Table 1. Summary of Parameters for Tremor Model.

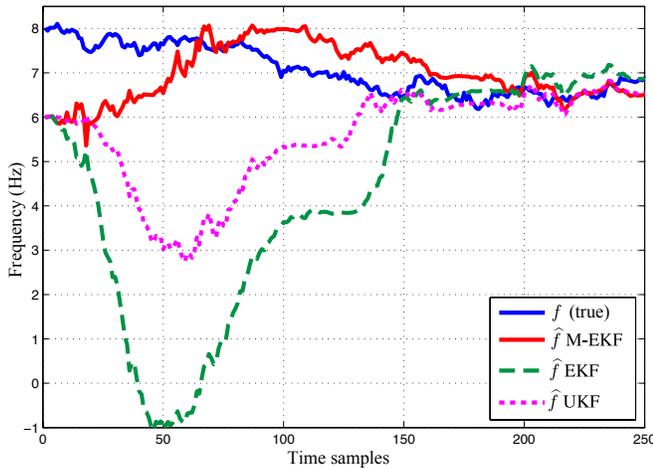
Name	Symbol	Value
Frequency variance	$q$	0.006
Measurement variance	$r$	0.6
Frequency bandwidth control parameter	$\gamma$	0.9987
Sampling interval (seconds)	$T_s$	1/1000
Amplitude of measurement sinusoid	$a$	$\sqrt{2}$
Mean frequency value (Hz)	$\bar{f}$	6
Mean theta value (radians)	$\bar{\theta}$	0
Initial state covariance	$\mathbf{P}(0 0)$	2I

## 4.2. Performance Criterion

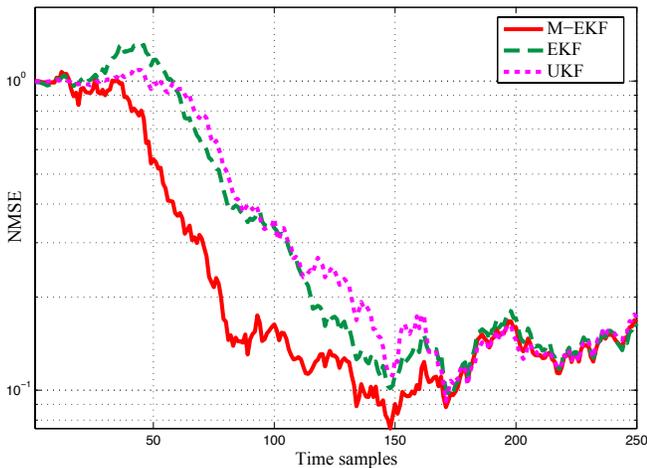
To adhere to the work of [3, 7], we use the normalized mean-square-error (NMSE) as the performance metric to measure the accuracy of the M-EKF frequency tracker.  $NMSE(k)$  is defined as

$$NMSE(k) = \frac{1}{MC} \sum_{i=1}^{MC} \frac{(f(k) - \hat{f}(k|k))^2}{(f(k) - \bar{f})^2}, \quad (13)$$

where  $f(k)$  is the true frequency,  $\hat{f}(k|k)$  is the estimated frequency, and  $\bar{f}$  is the mean frequency. MC is the number of Monte-Carlo simulations. An  $NMSE(k)$  value greater than one reflects that the accuracy of the frequency tracker is worse than a basic mean estimator, whereas a  $NMSE(k)$  value of zero implies exact tracking of the true frequency [7].



(a) Example of ITF tracking for a single mismatched realization.



(b) Averaged NMSE performance of the M-EKF versus the EKF and UKF.

**Fig. 5.** Performance of the M-EKF compared to the EKF and UKF.

## 4.3. Results

To generate synthetic tremor recordings, 100 random realizations of  $\mathbf{x}(0)$  were drawn from  $\mathcal{N}(\hat{\mathbf{x}}(0|0), \mathbf{P}(0|0))$  and propagated through the *state-space* model described in equations (11) and (12).

Figure 5(a) presents a single data track with mismatched realization  $\mathbf{x}(0)$  plotted against the M-EKF, EKF, and UKF estimates. The M-EKF performed noticeably better than the EKF and UKF, particularly in the early stages of tracking. The averaged  $NMSE(k)$  plots featured in Figure 5(b), demonstrate the performance of the three filters averaged over 100 Monte Carlo simulations.

Although the state equation (11) is linear, the observation equation (12) is highly nonlinear with the observation of the instantaneous phase  $\theta(k)$  through a sinusoid. The EKF introduces non-negligible linearization errors in such situations due to the poor representation of such nonlinearities through first-order Taylor series. The M-EKF performance eventually merges with that of the EKF. This is expected as the effect of initial conditions fades as time progresses, reducing the M-EKF into a bank of extended Kalman filters with equal weights and identical outputs. At this point, it is feasible to collapse the bank of EKFs and continue with a single EKF.

Comparison the UKF and EKF in Figure 5(b) shows that, on average, the two perform similarly. The study of [8] explains the unsatisfactory performance of the UKF in *state-space* models with non-scalar state vectors, such as this tremor model.

## 5. CONCLUSIONS

This paper proposed a new stochastic filter, the multiple extended Kalman filter. The M-EKF uses a bank of EKFs initialized with different initial conditions to mitigate the adverse effects of mismatches between assumed initial conditions of the filter and a true realization of a *state-space* model. As a specific application, the M-EKF was applied to the estimation of instantaneous tremor frequency in MERs. These observations demonstrate the potential of the M-EKF in decreasing the effects of initial condition mismatches in nonlinear tracking problems. Future work will include experimentation with clinical data and comparisons against other estimation solutions.

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